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The Application of Game Theory to Statistical Sampling Plans

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Abstract

The application of game theory to the design of Material, Control and Accounting (MC&A) sampling plans is discussed. Game theory can be applied to situations where there is an interdependency among the relevant decision-makers. In the context of MC&A sampling plans this interdependency could include relationships between the sampling plan designer and the regulator, the sampling plan designer and the facility/operations group, or the sampling plan designer and a postulated diverter of nuclear material. The game structure is formulated by considering sampling plans options set against the choices of an decision-maker. The optimal strategies between different interdependent relationships are examined in terms of choosing a sampling plan. To illustrate the optimization of the frequency of sampling, a payoff table between a sampling plan designer and a postulated diverted is constructed. Finally, an example of sensitivity analysis is conducted to illustrate the dependency of a sampling plan equilibrium to assumed payoffs.

Introduction

Game theory has become a fundamental tool in understanding economic and social phenomena, yet with some notable exceptions¹ game theory does not appear to be applied to routine domestic Material, Protection and Accountability (MC&A) practices, such as statistical sampling plans. This paper, by presenting the fundamentals of game theory as applied to statistical sampling plans, attempts to shed light on an analysis tool that can assist in the optimization of safeguards resources.

Investigations concerning the application of game theory and MC&A can involve a relatively high degree of complexity. Unfortunately, this paper risks, by the simplified examples presented, trivializing either MC&A or game theory. The author recognizes that the any application of game theory to MC&A will require a higher degree of sophistication than is presented in this paper; however, this fundamental presentation of the fundamental approach may be helpful the those readers who wish is understand the general context of game theory within the MC&A field.

Fundamentals of Game Theory

Game theory encompasses those situations in which decision-makers must take into account the decisions of others. Nevertheless, designing a statistical sampling plan may not fall under the above broad scope of game theory if the sampling plan designer (called the “designer” in the remainder of the paper) is only interested in meeting a performance specification or set of regulations. If, however, the actions of the approval body (called regulator in the remainder of the paper) of the sampling plan or any postulated diversion

of the nuclear material affect the actions of the designer, then the application of game theory may be able to assist the designer in developing the optimal sampling plan.

Knowledge of the presence of an entity that affects the choices of a decision-maker does not necessarily mean that the other independent decision-makers involved in the “game” are automatically “adversaries” whose choices must be opposed. Cooperative agreements are possible with some entities but not with others. For example, game theory could consider that both the following are games: (1) the designer taking into account the regulatory posture prior to submission of a sampling plan to a regulator and (2) the designer considering the options of a potential diversion prior to finalizing a sampling plan. Of course, in the case of the designer/regulator interaction cooperation may be possible, while a designer/diversion interaction will not likely incorporate cooperation or bargaining.

Games can be considered static or dynamic. Static games assume that the players move simultaneously or without knowledge of the opponent. An MC&A game can be considered static if the designer’s choice of sampling plans is unknown to the entity performing the diversion (called the “diverter”). If the diverter has knowledge of the sampling plan prior to any diversion, then the game can be considered dynamic. Dynamic games, where the moves are sequential, may possess outcomes and optimal solutions that are different than static games. In many situations, the player that moves first, in a dynamic game, can be at a disadvantage compared to a static game. For example, if a diverter is aware of the sampling plan that is to be implemented, then the diverter can choose a diversion strategy that will optimize his probability for nondetection to a greater degree than a static game would allow.

Games can further be divided into complete and incomplete games. Complete knowledge games assume a knowledge of the choices and the associated payoffs among decision-makers. Incomplete games, which do not make these assumptions, usually contain additional complexity. For example, if the sampling plan designer does not understand the diverter’s benefit of obtaining the nuclear material versus the loss of being detected, additional considerations are required to be evaluated. In general, the solution to a game problem with incomplete information will involve the use of Bayesian or other predictive methods. For example, the designed sample plan may be different depending on whether the diverter is assumed to be not worried of being caught or very afraid of being detected. Such an assumption for the diverter’s perceived “negative payoff” may be derived from historical data.

Discussion

A sampling plan is a primary material accountability method that demonstrates, with a degree of confidence, that nuclear material is present in the stated quantities and that physical controls are functioning. In constructing a statistical sampling plan the designer of the plan may be required to define a confidence interval (probability of nondetection) and minimum detectable defect (number of events in the population that have to occur for a given confidence interval). The sampling plan designer may have the option

of stratifying the population according the location, category level, measurement type and inventory frequency.

The choices confronting the plan developer may appear difficult to reconcile. If the designer optimizes the sampling plan for operational and regulatory requirements, has the plan been automatically optimized for detection of a diversion? By applying appropriate models and understanding the underlying assumptions the sampling plan developer is in a better position to coordinate the various options. The below game theory examples illustrate that just as players must form beliefs about how their opponents will play, they must also form beliefs about which game they are playing.

Example 1: Sampling by location

The sampling plan will be conducted at two locations: floor and vault. All the items are of similar category. Each area has 500 items. Material control is approximately the same for both areas. The regulatory goal is 95% confidence for a 3% defect. The facility goal is to minimize the impact sampling has on facility operations (minimize number of items sampled). The time permitted for the unified sampling plan is 100 samples and the time for the separate location-sampling plan is 70 for each location. The difference between sampling with and without replacement is considered negligible; therefore the relationship among nondetection probability (confidence level), number of samples and percent defect is the following,

$$d^n = \beta \quad (1)$$

where,

β = nondetection probability,
 n = number of samples,
 N = the total population size, and
 d = percentage of population that is nondefective.

The inspector has a choice of conducting two sample plans: Plan U randomly samples 100 items from a population of 1000. Plan V/F randomly samples two sets of 70 items from two respective populations of 500. Which plan should be chosen? Plan U, the unified sampling approach, provides a nondetection probability of 4.76 % (95.2% confidence) for a 3% defect rate. Plan V/F provides a nondetection probability of 16.1% (83.9% confidence). From a game perspective, the plan operator (who can influence the designer) desires to implement the most cost-effective plan; the regulator desires to implement the plan that possesses the highest nondetection probability. (given a fixed percent defect). The designer desires to have to meet the goals of operator and the regulator. The choices of the regulator and designer can be modeled as follows, with the bold indicating the optimal decision.

Regulator

<i>Designer</i>		Approve Plan	Disapprove Plan (approve other plan)
	Implement Plan U	100 samples/ $\beta=4.76\%$	120 samples/ $\beta=11.86\%$ + increased cost for negotiation
	Implement Plan V/F	120 samples/ $\beta=11.86\%$	100 samples/ $\beta=4.76\%$ + increased cost for negotiation

Now consider the same constraints but that the “opponent” is the diverter of the material. The diverter is interested the diversion of a single from any location, the formula then can be written as

$$(1-10/N)^n = \beta \quad (2)$$

Assume that the vault has, due to augmented material control an additional 50% detection probability against a diversion. Using the same parameters and optimization criteria for the auditor as above, the matrix and saddle point can be written as:

Diverter

<i>Designer</i>		Divert for floor	Divert from vault
	Implement Plan A	100 samples/ $\beta=$ 36.60%	100 samples/ $\beta=18.30\%$
	Implement Plan F/V	140 samples/ $\beta=24.31\%$	140 samples/ $\beta=12.16\%$

By the designer optimizing for regulatory approval and sample size, the diverter is able to select the cell that globally maximizes the nondetection probability. If the designer had optimized by taking onto account the diverter’s point-of-view then Plan F/V would have been implemented.

Example 2: Sampling by Category

Assume that the designer is required to stratify by material category. One location is being designed for a sample plan. The parameters are 95% confidence with a 3% defect for high category items and 95% confidence with 10% defect for low category items. The location has 1000 items with 500 items in each category. Operations proposes a plan to the designer that stratified sampled will not be perform sampling by stratification but that the 10% addition samples that will be taken to the calculated sample size that meets the criteria for a unified population. Plan S, the stratified plan, will require 99 samples for the high category material and 29 for the low-category material. Plan U, the unified plan, will require 128 samples for the entire population. Assume that regulator desires the plan that possesses the smallest nondetection probability and that the designer prefers the plan that is preferred by operations. The choices of the regulator and designer can be modeled as follows, with the bold indicating the optimal decision.

		<i>Regulator</i>	
<i>Designer</i>		Approve Plan	Disapprove Plan (approve other plan)
	Implement Plan U	128 samples/ $\beta=2.03\%$ (all 3% defect)	128 samples/ $\beta_{\text{high}}=4.90\%$ (3% defect) $\beta_{\text{low}}=4.71\%$ (10% defect)+ increased cost for negotiation
	Implement Plan S	128 samples/ $\beta_{\text{high}}=4.90\%$ (3% defect) $\beta_{\text{low}}=4.71\%$ (10% defect)	128 samples/ $\beta=2.03\%$ +increased cost for negotiation

If the diverted is now considered the "opponent" where the diverter is interested in diverting one item of high-category material or 3 items of low-category material, then the game matrix would be:

		<i>Diverter</i>	
<i>Designer</i>		Divert 1 high-category	Divert 3 low-category
	Implement Plan A	128 Samples/$\beta=88\%$	128 samples/ $\beta=68\%$
	Implement Plan C	128 Samples/ $\beta=82\%$	128 samples/ $\beta=84\%$

By the auditor optimizing for overall nondetection probability and sample size, the diverter is able to select the cell that, again globally optimizes his payoff.

Example 3: Sampling by category or location

The auditor has the choice of two distinct stratification sampling plans. The sampling plan can be stratified by location or material category. Assume that the vault and the floor each have 600 and 400 items respectively and that 700 of the items in the two areas are high category and 300 are low-category. The vault provides a 50% chance that an unauthorized activity will be detected. The diverter will take this extra rule into account, but the designer will not. The material is in two quantities "category-high" and 'category-low'. The auditor can either choose a plan (both with the same resource requirements) based on stratification by category (Plan C) or by location (Plan L). Plan C has two sampling plans which provide for both 95% confidence for a 3% defect for "category-high" material (99 samples) and 10% defect for 'category-low" material (29 samples), regardless of location. Plan L has two sampling plans that provide for a 95% confidence and a 3% defect rate (99 samples for each location) for both the vault and the floor. The auditor recommends Plan C to the regulator (since Plan C has less samples) Assume that the auditor will approve plan C (since Plan C is required by the regulations). The game matrix between the regulator and the saddle point are shown below.

		<i>Regulator</i>	
<i>Designer</i>		Approve Plan	Disapprove Plan (approve other plan)
	Implement Plan L	198 samples/ $\beta=4.90\%$ (all 3% defect)	128 samples/ $\beta_{\text{high}}=4.90\%$ (3% defect) $\beta_{\text{ow}}=4.71\%$ (10% defect)+ increased cost for negotiation
	Implement Plan C	128 samples/ $\beta_{\text{high}}=4.90\%$ (3% defect) $\beta_{\text{low}}=4.71\%$ (10% defect)	128 samples/ $\beta=2.03\%$ +increased cost for negotiation

If the diverted is now considered the "opponent" where the diverter is interested in diverting one item of high-category material or 3 items of low-category material, then the game matrix would be:

		<i>Diverter</i>			
<i>Designer</i>		Divert from floor high-category	Divert from floor low-category	Divert from vault high-category	Divert from vault low-category
	Plan L (198 samples)	78.1%	47.5%	84.8%	60.1%
	Plan C (128 samples)	86.8%	74.7%	86.8%	74.7%

Note that the optimal choice for the designer and regulator, becomes the optimal choice for diverter.

Example 4: Choosing the plan with more samples

Using the same models in example 3, but assume that 500 items are in each category and 700 items on the floor and 300 items in the vault. Assume that the regulator chooses to disapprove Plan C and approve Plan L, for two reasons: Plan C has more samples and the regulator desires a uniform statistical parameters for any location; therefore Plan L is chosen. The designer/diverter model will be as follows:

Diverter

		Divert from floor high- category	Divert from floor low- category	Divert from vault high- category	Divert from vault low- category
<i>Diverter</i>	Plan L (198 samples)	86.8%	65.4%	71.9%	35.0%
	Plan C (128 samples)	82.0%	55.1%	82.0%	55.1%%

Again, the designer , by not considering the diverter's point-of-view has allowed the diverter to globally optimize his payoff.

Example 5: Sample plan design using game theory

A relatively simple method of optimizing a sample plan would be to set all the nondetection probabilities equal and solve for the number of samples. Assume that the population in each of the four category/location combinations are as follows: floor-high: 350, floor-low: 350, vault-high 150, vault-low 150. Assuming a fixed number of samples (198), so that the four categories are equal.

where:

n1=number of samples taken from floor-high

n2=number of samples taken from floor-low

n3=number of samples taken from vault-high

n4=number of samples taken from vault-low

$$(1-1/350)^{n1}=(1-3/350)^{n2}=(1-1/150)^{n3}=(1-3/150)^{n4} \text{ and}$$

$n1+n2+n3+n4=198$ Rounding to the nearest integer, the solution to the above equations are $n1=104$, $n2=35$, $n3=44$ and $n4=15$, providing for guaranteed nondetection probability of approximately 74%, lower than any previously analyzed saddle point. If the additional material control of the vault could be taken into account, the nondetection probability will be even lower.

Example 6: Optimizing an inspection frequency

The frequency of an audit can be optimized by considering the diverter's point-of-view. Consider that each month the facility can decide to conduct a sampling plan or not. The expenditure of resources without the detection is considered a slight negative for the facility (and a positive for a diverter). Assume the following payoff matrix (The payoff of the diverter is the opposite sign of the value given in the matrix):

Diverter

<i>Designer</i>		Divert	Do not Divert
	No sample (no detection of diversion)	$a = -50/\text{month}$	$b = 5/\text{month}$
	Sample (detect diversion)	$c = 20/\text{month}$	$d = -1/\text{month}$

The above matrix does not possess a pure strategy. In such cases where no single choice is dominant, then a particular mixed strategy can be shown to be the “saddlepoint” of the game. The equilibrium strategy for the designer (maximin) can be calculated as $\{\text{row 1, row 2}\} = \{(c-d)/(c-d+b-a), (b-a)/(c-d+b-a)\} = (21/76, 55/76)$; therefore, considering the above payoff matrix, the designer should sample in about two-thirds of the months. The expected payoff for such a strategy (maximin value) would be $(bc-ad)/(c-d+b-a) = 50/76$. As an example of sensitivity analysis, consider the penalty for a diversion occurring in month of no sampling to be quadrupled to $-200/\text{month}$; the equilibrium strategy would change to $(21/226, 205/226)$; indicating that an increased sampling frequency would be optimal. The expected payoff for such a strategy (maximin value) would be $-100/76$.

Summary

The application of game theory to MC&A sampling plans provides sampling plan designers with an analytical framework to analyze decisions in the context of regulatory, operational and diversionary choices. Sampling plan options that are optimized in a regulatory structure are not necessarily optimal in terms of detecting a diversion. For cases where a pure strategy or a dominant sampling plan does not exist, the application of game theory can determine a defensible mixed strategy, which guarantees an expected payoff. Finally, the application of game theory provides the framework for both sensitivity analysis and the modification of imperfect knowledge.

References

- [1] Avenhaus, Rudolf, “Material Accountability: Theory, Verification and Applications”, International Institute for Applied Systems Analysis, 1977.
- [1] Stahl, Saul, “A Gentle Introduction to Game Theory”, Mathematical World; v.13 1998
- [2] Beirman, H. Scott, “Game Theory with Economic Applications”, 2nd ed.; Addison – Wesley 1998